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3D-Similarity Transformation related to (B,L,h) – Molodenski-Approach

- Concept and Advantages

We start with the general relation between the geocentric cartesian coordinates (X,Y,Z) and the geographical coordinates (B,L,h) referring to a reference ellipsoid with given axes (a,b):

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} (N(B)+h)\cdot\cos(B)\cdot\cos(L) \\ (N(B)+h)\cdot\cos(B)\cdot\sin(L) \\ (\frac{b^2}{a^2}\cdot N(B)+h)\cdot\sin(B) \end{bmatrix} \quad . \tag{1}$$

By the differentiation of (1) we get:

$$\begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial B} & \frac{\partial X}{\partial L} & \frac{\partial X}{\partial h} \\ \frac{\partial Y}{\partial B} & \frac{\partial Y}{\partial L} & \frac{\partial Y}{\partial h} \\ \frac{\partial Z}{\partial B} & \frac{\partial Z}{\partial L} & \frac{\partial Z}{\partial h} \end{bmatrix} \cdot \begin{bmatrix} dB \\ dL \\ dh \end{bmatrix}$$

$$\begin{bmatrix} -(M+h)\cdot\sin(B)\cdot\cos(L) & -(N+h)\cdot\cos(B)\cdot\sin(L) & \cos(B)\cos(L) \\ -(M+h)\cdot\sin(B)\sin(L) & (N+h)\cdot\cos(B)\cdot\cos(L) & \cos(B)\sin(L) \\ (M+h)\cdot\cos(B) & 0 & \sin(B) \end{bmatrix} \cdot \begin{bmatrix} dB \\ dL \\ dh \end{bmatrix} \tag{2}$$

$$= [A(dX\_dGeo)] \cdot \begin{bmatrix} dB \\ dL \\ dh \end{bmatrix}$$

For two points $(X,Y,Z)_1$ and $(X,Y,Z)_2$ situated in a differential neighbourhood, we can apply (2) and write:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 - \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 = \begin{bmatrix} A(dX\_dGeo) \end{bmatrix}_1 \cdot \left(\begin{bmatrix} B \\ L \\ h \end{bmatrix}_2 - \begin{bmatrix} B \\ L \\ h \end{bmatrix}_1 \right) \quad . \tag{3}$$

The index 1 indicates, that the matrix **A**(dX_dGeo) is to be computed as a linearization referring to the linearization point $(B,L,h)_1$. By the inversion of (3) we arrive at:

$$\begin{bmatrix} B \\ L \\ h \end{bmatrix}_2 - \begin{bmatrix} B \\ L \\ h \end{bmatrix}_1 = \begin{bmatrix} & A(dX\_dGeo) & \end{bmatrix}_1^{-1} \cdot \left(\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 - \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 \right) \quad , \quad \text{with} \qquad (4a)$$

$$\begin{bmatrix} & A(dX\_dGeo) & \end{bmatrix}_1^{-1} = \begin{bmatrix} \frac{-\sin(B)\cdot\cos(L)}{(M+h)} & \frac{-\sin(B)\sin(L)}{(M+h)} & \frac{\cos(B)}{(M+h)} \\ \frac{-\sin(L)}{(N+h)\cdot\cos(B)} & \frac{\cos(L)}{(N+h)\cdot\cos(B)} & 0 \\ \cos(B)\cos(L) & \cos(B)\sin(L) & \sin(B) \end{bmatrix} \quad (4b)$$

In the following we asssume, that the two differential positions $(X,Y,Z)_1$ and $(X,Y,Z)_2$ above are due to a (…. differential ….) datum transition, which is to be described with set of datum parameters $\mathbf{d} = (\varepsilon_x,,\varepsilon_y,\varepsilon_z,s,t_x,t_y,t_z)$. The parameter set **d** comprises 3 rotations round the axis $\mathbf{d}_{rot} = (\varepsilon_x,\varepsilon_y,\varepsilon_z)$, one scale d_s=s and three translations $\mathbf{d}_{trans} = (t_x,t_y,t_z)$.

Using the classical relations of a non-linear seven parameter similarity transformation for the above datum transition we arrive at:

$$\begin{bmatrix} B \\ L \\ h \end{bmatrix}_2 = \begin{bmatrix} B \\ L \\ h \end{bmatrix}_1 + \begin{bmatrix} & A(dX\_dGeo) & \end{bmatrix}_1^{-1} \cdot \left(s \cdot R(\varepsilon_x,\varepsilon_y,\varepsilon_z) \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 + t - \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 \right). \quad (5)$$

According to (2) we have to require that all datum parameters – also the translations[1] - are small quantities, so that (5) may be linearized a second time, now at the linearization point $\mathbf{d}_0 = (\varepsilon_x = 0, \varepsilon_y = 0, \varepsilon_z = 0, s = 1, t_x = 0, t_y = 0, t_z = 0)$. Under that assumptions we arrive at:

[1] This leads to the neccessity that a preceeding translatorial pre-transformation has to be applied to system 1 in case of large translations. But as translation parameters in the necessary accurracy range of some meters are available for all datum transitions round the world this requirement does not disturb in practice.

$$\begin{bmatrix} B \\ L \\ h \end{bmatrix}_2 = \begin{bmatrix} B \\ L \\ h \end{bmatrix}_1 - \Big[\; A(dX\_dGeo) \; \Big]_1^{-1} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 +$$

$$\Big[\; A(dX\_dGeo) \; \Big]_1^{-1} \left(\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 + \left[\begin{array}{ccc|c|ccc} 0 & -Z_1 & Y_1 & X_1 & 1 & 0 & 0 \\ Z_1 & 0 & -X_1 & Y_1 & 0 & 1 & 0 \\ -Y_1 & X_1 & 0 & Z_1 & 0 & 0 & 1 \end{array} \right] \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \Delta s \\ t_x \\ t_y \\ t_z \end{bmatrix} \right) \qquad (6a)$$

and

$$\begin{bmatrix} B \\ L \\ h \end{bmatrix}_2 = \begin{bmatrix} B \\ L \\ h \end{bmatrix}_1 + \Big[\; A(dX\_dGeo) \; \Big]_1^{-1} \cdot \left(\left[\begin{array}{ccc|c|ccc} 0 & -z_1 & y_1 & x_1 & 1 & 0 & 0 \\ z_1 & 0 & -x_1 & y_1 & 0 & 1 & 0 \\ -y_1 & x_1 & 0 & z_1 & 0 & 0 & 1 \end{array} \right] \right) \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \Delta s \\ t_x \\ t_y \\ t_z \end{bmatrix} \qquad (6b)$$

respectively. In (6b) we can again introduce the relations (1) between the cartesion coordinates $(X,Y,Z)_1$ and the den geographical coordinates (B,L,h), and we arrive after the complete multiplication of the two matrix expressions at the following final formulas ([1]- [6]):

$$\begin{bmatrix} B \\ L \\ h \end{bmatrix}_2 = \begin{bmatrix} B \\ L \\ h \end{bmatrix}_1 + \big[\text{Moldenski}\big]_{(B,L,h)_1} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \Delta s \\ t_x \\ t_y \\ t_z \end{bmatrix} \qquad (7a)$$

For the Molodenski-matrix we get from (6b) on introducing (4b) and (1) there

$$[\text{Moldenski}]_{(B,L,h)_1} =$$

$$\begin{bmatrix} -\sin(L)\cdot\frac{a\cdot W+h}{M+h} & \cos(L)\cdot\frac{a\cdot W+h}{M+h} & 0 & \frac{-\sin(B)\cdot\cos(B)\cdot N\cdot e^2}{M+h} & \frac{-\sin(B)\cdot\cos(L)}{M+h} & \frac{-\sin(B)\cdot\sin(L)}{M+h} & \frac{\cos(B)}{M+h} \\ \frac{\sin(B)\cdot\cos(L)\cdot(N\cdot(1-e^2)+h)}{(N+h)\cdot\cos(B)} & \frac{\sin(B)\cdot\sin(L)\cdot(N\cdot(1-e^2)+h)}{(N+h)\cdot\cos(B)} & -1 & 0 & \frac{-\sin(L)}{(N+h)\cdot\cos(B)} & \frac{\cos(L)}{(N+h)\cdot\cos(B)} & 0 \\ -N\cdot e^2\cdot\sin(B)\cdot\cos(B)\cdot\sin(L) & N\cdot e^2\cdot\sin(B)\cdot\cos(B)\cdot\cos(L) & 0 & h+a\cdot W & \cos(B)\cdot\cos(L) & \cos(B)\cdot\sin(L) & \sin(B) \end{bmatrix}$$

(see appendix) (7b)

with

$$W = \frac{a}{N} = \sqrt{1-e^2\cdot\sin^2 B} \quad \text{and} \quad e^2 = \frac{a^2-b^2}{a^2} \quad .$$

In case however, that the position $(B,L,h)_1$ is related both to a different datum and to another ellipsoid - meaning that the positions $(B,L,h)_1$ and $(B,L,h)_2$ belong to different reference ellipsoids (a_1,b_1) und (a_2,b_2) - we have to take into account for the transition to $(B,L,h)_2$ the additonal corrections $(\Delta B,\Delta L,\Delta h)_{(a,b)_1,(a,b)_2}$ of a so-called ellipsoid transition.

So we get the final formulas for that general case, as they have been implemented in the software packages COPAG [7] and WTRANS [7], the following oberservation equations for observed point positions (B,L,h) in two reference frames:

$$\left(\begin{bmatrix} B \\ L \\ h \end{bmatrix}_2 - \begin{bmatrix} \Delta B_{(a,b)_1,(a,b)_2} \\ \Delta L_{(a,b)_1,(a,b)_2} \\ \Delta h_{(a,b)_1,(a,b)_2} \end{bmatrix} - \begin{bmatrix} B \\ L \\ h \end{bmatrix}_1 \right)_i + \begin{bmatrix} v_B \\ v_L \\ v_h \end{bmatrix}_i = [\text{Moldenski}]_{(B,L,h)_1,i} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \Delta s \\ t_x \\ t_y \\ t_z \end{bmatrix} \qquad (7c)$$

with

$$\Delta B_{(a_1,b_1),(a_2,b_2)} = B(a_2,b_2 \mid (X,Y,Z)_1) - B(a_1,b_1 \mid (X,Y,Z)_1)\ ,$$

$$\Delta L_{(a_1,b_1),(a_2,b_2)} = 0 \ \text{and}$$

$$\Delta h_{(a_1,b_1),(a_2,b_2)} = h(a_2,b_2 \mid (X,Y,Z)_1) - h(a_1,b_1 \mid (X,Y,Z)_1)\ .$$

Discussion

The advantage of the system of observation equations (functional model) as given by (7b) compared to the similarity transformation in cartesion coordinates (X,Y,Z) is, that (7b) can simultaneously be used for

- complete 3D identical points (B,L,h) in both systems, as well as for
- pure 2D identical horizontal points (B,L) in both systems, and for
- pure 1D identical height points (h) in both systems.

Alternative

Alternatively to (7b) the non-linear datum transformation in cartesion coordinates

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = s \cdot R(\varepsilon_X, \varepsilon_Y, \varepsilon_Z) \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 + t \tag{8a}$$

as well as the linearized version

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_2 = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_1 + \left[\begin{array}{ccc|c|ccc} 0 & -z_1 & y_1 & x_1 & 1 & 0 & 0 \\ z_1 & 0 & -x_1 & y_1 & 0 & 1 & 0 \\ -y_1 & x_1 & 0 & z_1 & 0 & 0 & 1 \end{array}\right]_1 \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \Delta s \\ t_x \\ t_y \\ t_z \end{bmatrix} \tag{8b}$$

can be used in case of such a mixed situation of identical points by creating a stochastical model, which is adapted to the respective uncertainties. With

$$\begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial B} & \frac{\partial X}{\partial L} & \frac{\partial X}{\partial h} \\ \frac{\partial Y}{\partial B} & \frac{\partial Y}{\partial L} & \frac{\partial Y}{\partial h} \\ \frac{\partial Z}{\partial B} & \frac{\partial Z}{\partial L} & \frac{\partial Z}{\partial h} \end{bmatrix} \cdot \begin{bmatrix} dB \\ dL \\ dh \end{bmatrix} = F \cdot \begin{bmatrix} dB \\ dL \\ dh \end{bmatrix} \tag{2}$$

an appropriate stochastical model

$$C_{X,Y,Z} = F \cdot \begin{bmatrix} \sigma_B^2 & & \\ & \sigma_L^2 & \\ & & \sigma_h^2 \end{bmatrix} \cdot F^T \tag{9}$$

for the cartesion coordinates (X,Y,Z) can be evaluated for each identical point according to its uncertainities in (B,L) (large values for σ_B and σ_L) in case of an 1D identical point, or its uncertainties in (h) (large value for σ_h in case of 2D identical point).

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Appendix: Molodenski-Transformation and Molodenski-Matrix (7b)

$$\begin{bmatrix} B_2 \\ L_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ L_1 \\ h_1 \end{bmatrix} + [\mathrm{Molodensik}]_{(B,L,h)_1} \cdot \hat{\mathbf{x}}$$

$$[\mathrm{Molodensik}]_{(B,L,h)_1} =$$

$$\begin{bmatrix} -\sin(L)\cdot\dfrac{a\cdot W+h}{M+h} & \cos(L)\cdot\dfrac{a\cdot W+h}{M+h} & 0 & \dfrac{-\sin(B)\cdot\cos(B)\cdot N\cdot e^2}{M+h} & \dfrac{-\sin(B)\cdot\cos(L)}{M+h} & \dfrac{-\sin(B)\cdot\sin(L)}{M+h} & \dfrac{\cos(B)}{M+h} \\ \dfrac{\sin(B)\cdot\cos(L)\cdot(N\cdot(1-e^2)+h)}{(N+h)\cdot\cos(B)} & \dfrac{\sin(B)\cdot\sin(L)\cdot(N\cdot(1-e^2)+h)}{(N+h)\cdot\cos(B)} & -1 & 0 & \dfrac{-\sin(L)}{(N+h)\cdot\cos(B)} & \dfrac{\cos(L)}{(N+h)\cdot\cos(B)} & 0 \\ -N\cdot e^2\cdot\sin(B)\cdot\cos(B)\cdot\sin(L) & N\cdot e^2\cdot\sin(B)\cdot\cos(B)\cdot\cos(L) & 0 & h+a\cdot W & \cos(B)\cdot\cos(L) & \cos(B)\cdot\sin(L) & \sin(B) \end{bmatrix}$$

Parameter-Vector $\hat{\mathbf{x}} = [\varepsilon_x\,[\mathrm{rad}] \quad \varepsilon_y\,[\mathrm{rad}] \quad \varepsilon_z\,[\mathrm{rad}] \quad \Delta s\,[\text{-}] \quad t_x\,[\mathrm{m}] \quad t_y\,[\mathrm{m}] \quad t_z\,[\mathrm{m}]]^T$ [2]

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[2] The Molodenski approach requires infinite (small) parameters meaning in that case also small translations. Finite translations therefore require a deterministic translational pre-transformation applied to all positions $\mathbf{x_1}$. This leads to $\mathbf{x_1}' = [x_1', y_1', x_1']^T = \mathbf{x}_1 + \mathbf{x}_0$ using approximate parameters, e.g. for the transformation of the classical DHDN-datum Germany to ETRS89 reading $\mathbf{x}_0 = [0, 0, 0 \,|\, 0 \,|\, 596\mathrm{m}, 72\mathrm{m}, 414\,\mathrm{m}]^T = [\varepsilon_{x,0}, \varepsilon_{y,0}, \varepsilon_{z,0} \,|\, \Delta s_0 \,|\, t_{x,0}, t_{y,0}, t_{z,0}]$. Approximate translation parameters $t_0 = [t_{x,0}, t_{y,0}, t_{z,0}]$ are published for most all countries, e.g. easily found in the internet, or can be derived direct form the data in different ways. From $\mathbf{x_1}' = [x_1', y_1', x_1']^T$ the pre-transformed positions (B_1', L_1', h_1') are to be computed. The(B_1', L_1', h_1') are already „close" to the datum 2, and in that way "differentially" (in a range of e.g. 10-20 m close to (B_2, L_2, h_2). Only the transformed positions (B_1', L_1', h_1') are to be used as basic input from datum to all above Molodenski formulas and computations to come out with strict and undistorted final results.